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Entropy Numbers of Diagonal Operators with an Application to Eigenvalue Problems

BERND CARL

Sektion Mathematik, Universität Jena, DDR 69 Jena, GDR

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In this paper we characterize diagonal operators from l_p into l_q, $1 \leqslant p, q \leqslant \infty$, by their entropy numbers. The results contain those of Marcus [16] formulated in the language of ε-entropy and Oloff [21]. Moreover, the remaining gaps in [16, 21] are filled in the much more complicated situation where $1 \leqslant p < q \leqslant \infty$.

Furthermore, we extend the results to diagonal operators acting between Lorentz sequence spaces. It turns out that the computation of entropy numbers of diagonal operators can be reduced to the computation of certain entropy quasi-ideal norms of identity operators acting between the simple n-dimensional vector spaces l_p^n.

Finally, the entropy numbers are used for studying eigenvalue problems of factorable operators acting on Banach spaces.

The statements of this paper are obtained using results and techniques recently proved and developed by the author in [4].

TERMINOLOGY

In the sequel almost all notations and some basic definitions are adopted from [24]. Concerning interpolation theory we refer to [1, 27].

The class of all (bounded linear) operators between arbitrary Banach spaces is denoted by $\mathscr{L}$, while $\mathscr{L}(E, F)$ stands for the set of those operators acting from E into F.

If $x = (\xi_k)$ is a bounded sequence, then we put

$$s_n(x) := \inf\{\sigma \geqslant 0\colon \operatorname{card}(k\colon |\xi_k| \geqslant \sigma) < n\}.$$

In these case $|\xi_1| \geqslant |\xi_2| \geqslant \cdots \geqslant 0$ it turns out that $s_n(x) = |\xi_n|$. Therefore $(s_n(x))$ is called the non-increasing rearrangement of x. For $0 < p, u \leqslant \infty$ the

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Lorentz sequence space $l_{p,u}$ consists of all sequences $x = (\xi_k)$ having a finite quasinorm

$$\|x\|_{p,n} := \left(\sum_{n=1}^{\infty} [n^{1/p - 1/u} s_n(x)]^u \right)^{1/u} \qquad \text{if} \quad 0 < u < \infty$$

$$:= \sup_{n=1,2,\ldots} [n^{1/p} s_n(x)] \qquad \text{if} \quad u = \infty.$$

For $p = u$ we get the classical space of p-summable sequences which is denoted by $[l_p, \| \|_p]$. The n-dimensional vector equipped with the norm $\| \|_p$, $1 \leqslant p \leqslant \infty$, is denoted by l_p^n.

The scale of Lorentz sequence spaces is ordered lexicographically. This means that

$$l_{p_1,u_1} \subset l_{p_2,u_2} \qquad \text{if} \quad 0 < p_1 < p_2 \leqslant \infty, \quad 0 < u_1, \quad u_2 \leqslant \infty,$$

and

$$l_{p,u_1} \subset l_{p,u_2} \qquad \text{if} \quad 0 < p \leqslant \infty, \quad 0 < u_1 < u_2 \leqslant \infty.$$

In the sequel, by $\rho(s, p,\ldots)$, $\rho_0(s, p,\ldots)$, $\rho_1(s, p,\ldots)$,... we always denote positive constants depending only on s, p,..., but not on the quantity n.

1. Entropy Numbers, s-Numbers, and Operator Ideals

For every operator $S \in \mathscr{L}(E, F)$ the nth entropy number $e_n(S)$ is defined to be the infimum of all $\varepsilon \geqslant 0$ such that there are $y_1, \ldots, y_{2^{n-1}} \in F$ for which

$$S(U_E) \subseteq \bigcup_{1}^{2^{n-1}} \{y_i + \varepsilon U_F\}$$

holds. The sets U_E and U_F are the closed unit balls of E and F, respectively.

Roughly speaking, the asymptotic behaviour of $e_n(S)$ characterizes the "degree of compactness" of S. In particular, S is compact if and only if $\lim e_n(S) = 0$.

The theory of entropy numbers was developed by Pietsch for the first time in [24, (12)]. However, certain functions inverse to the ε-entropy already appeared in Mitjagin and Pełczyński [19] and Triebel [28].

The concept of entropy numbers is related to that of ε-entropy first studied by Pontrijagin and Schnirelman [26] in 1932. Further contributions are mainly due to Soviet mathematicans (e.g., [2, 13]). For additional information the reader is referred to the monograph of Lorentz [15]; see also [14].

The entropy numbers have the following nice properties of an additive and multiplicative s-number function [24, (12)]:

monotonicity,

$$\|S\| = e_1(S) \geqslant e_2(S) \geqslant \cdots \geqslant 0 \qquad \text{for } S \in \mathscr{L}(E, F);$$

additivity,

$$e_{n+m-1}(S+T) \leqslant e_n(S) + e_m(T) \qquad \text{for } S, T \in \mathscr{L}(E, F);$$

multiplicativity,

$$e_{n+m-1}(ST) \leqslant e_n(S)\, e_m(T) \qquad \text{for } T \in \mathscr{L}(E, F), \quad S \in \mathscr{L}(F, G).$$

Put

$$\mathscr{L}_{s,t}^{(e)} := \{S \in \mathscr{L} \colon (e_n(S)) \in l_{s,t}\}$$

and

$$L_{s,t}^{(e)}(S) := \kappa_{s,t}\, \|(e_n(S))\|_{s,t} \qquad \textit{for } S \in \mathscr{L}_{s,t}^{(e)},$$

where $\kappa_{s,t}$ is some norming constant; then $[\mathscr{L}_{s,t}^{(e)}; L_{s,t}^{(e)}]$ becomes a quasinormed operator ideal [24], (14.3). We write $[\mathscr{L}_s^{(e)}; L_s^{(e)}]$ instead of $[\mathscr{L}_{s,s}^{(e)}; L_{s,s}^{(e)}]$. From the multiplicativity of the entropy numbers we get the useful product formula

$$\mathscr{L}_{s_2,t_2}^{(e)} \circ \mathscr{L}_{s_1,t_1}^{(e)} \subseteq \mathscr{L}_{s,t}^{(e)} \qquad \text{for } \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}, \quad \frac{1}{t} = \frac{1}{t_1} + \frac{1}{t_2}.$$

The definition of the product of quasinormed operator ideals comes from [24, (7.1)].

For our purpose we also need the following s-numbers: If $S \in \mathscr{L}(E, F)$ and $n = 1, 2, \ldots$, then the nth Gelfand number and Kolmogorov number are defined by

$$c_n(S) := \operatorname{ind}\{\|SJ_M^E\| \colon \operatorname{codim}(M) < n\}$$

and

$$d_n(S) := \inf\{\|Q_N^F S\| \colon \dim(N) < n\},$$

respectively. Here J_M^E is the embedding map from M into E and Q_N^F the canonical map from F onto F/N.

It should be mentioned that $c := (c_n)$ and $d := (d_n)$ are additive and multiplicative s-number functions [24, (11)]. The quasinormed operator ideals $[\mathscr{L}_{s,t}^{(c)}; L_{s,t}^{(c)}]$ and $[\mathscr{L}_{s,t}^{(d)}; L_{s,t}^{(d(}]$ generated by the Gelfand numbers and Kolmogorov numbers, respectively, are defined analogously as above.

2. Entropy Quasinorms of Identity Operators from l_p^n onto l_q^n

The aim of this section is to estimate the entropy quasinorms $L_{s,t}^{(e)}(I_n\colon l_p^n \to l_q^n)$ of the identity operators I_n from l_p^n onto l_q^n for $1 \leqslant p, q \leqslant \infty$ and $n = 1, 2,\ldots$. The estimates given below improve those of [6; 24, (14.4.12)], for $1 \leqslant p < q \leqslant \infty$, where an additional logarithmic term appears. For our purpose we need two known theorems. The first theorem is a result of [4].

THEOREM A. *Let* $0 < s \leqslant \infty$, *and* $0 < t \leqslant \infty$. *Then*

$$\mathscr{L}_{s,t}^{(c)}(E, F) \subseteq \mathscr{L}_{s,t}^{(e)}(E, F) \qquad \text{and} \qquad \mathscr{L}_{s,t}^{(d)}(E, F) \subseteq \mathscr{L}_{s,t}^{(e)}(E, F)$$

for all Banach spaces E and F.

The following very striking result has been proved by Kashin [11] and Mitjagin [18].

THEOREM B. *There exists a constant* $\rho > 0$ *such that*

$$c_k(I_n\colon l_1^n \to l_2^n) \leqslant \rho \left[\frac{\log^3(n/k+1)}{k}\right]^{1/2}$$

for $k = 1, 2,\ldots, n$ *and* $n = 1, 2,\ldots$.

Now we are able to prove the main statement of this section.

THEOREM 1. *Let* $1 \leqslant p, q \leqslant \infty$, $1/s > \max(1/p - 1/q; 0)$, *and* $0 < t \leqslant \infty$. *Then*

$$L_{s,t}^{(e)}(I_n\colon l_p^n \to l_q^n) \leqslant \rho(s, t, p, q)\, n^{1/s + 1/q - 1/p}$$

for $n = 1, 2,\ldots$.

Proof. By Lemma 1 of [4], for the identity operator I_n on an arbitrary n-

dimensional Banach space we have $L^{(e)}_{s,t}(I_n) \leqslant \rho(s,t)\, n^{1/s}$ for $n = 1, 2,\ldots$. Using this we obtain

$$\begin{aligned} L^{(e)}_{s,t}(I_n\colon l^n_p \to l^n_q) &\leqslant L^{(e)}_{s,t}(I_n\colon l^n_q \to l^n_q)\, \|I_n\colon l^n_p \to l^n_q\| \\ &\leqslant \rho(s,t)\, n^{1/s} n^{1/q - 1/p} \\ &\leqslant \rho(s,t)\, n^{1/s + 1/q - 1/p} \end{aligned}$$

for $1 \leqslant q \leqslant p \leqslant \infty$ and $n = 1, 2,\ldots$.

Now we turn to the more complicated situation, where $1 \leqslant p < q \leqslant \infty$. First we treat the case $p = 1$, $q = 2$. To do this let $0 < r < 2$. Theorem B yields

$$\begin{aligned} L^{(c)}_{r,\infty}(I_n\colon l^n_1 \to l^n_2) &= \sup_{1 \leqslant k \leqslant n} [k^{1/r} c_k(I_n\colon l^n_1 \to l^n_2)] \\ &\leqslant \rho_0 \sup_{1 \leqslant k \leqslant n} k^{1/r - 1/2} \log^{3/2}(n/k + 1) \\ &\leqslant \rho_1(r)\, n^{1/r - 1/2}. \end{aligned}$$

From Theorem A we have $\mathscr{L}^{(c)}_{r,\infty} \subseteq \mathscr{L}^{(e)}_{r,\infty}$ and therefore by [24], (6.1.6), there exists a constant $\rho_2(r)$ such that

$$L^{(e)}_{r,\infty}(S) \leqslant \rho_2(r)\, L^{(c)}_{r,\infty}(S) \qquad \text{for } S \in \mathscr{L}^{(c)}_{r,\infty}.$$

Hence

$$\begin{aligned} L^{(e)}_{r,\infty}(I_n\colon l^n_1 \to l^n_2) &\leqslant \rho_2(r)\, L^{(c)}_{r,\infty}(I_n\colon l^n_1 \to l^n_2) \\ &\leqslant \rho_3(r)\, n^{1/r - 1/2} \end{aligned}$$

for $0 < r < 2$ and $n = 1, 2,\ldots$.

Analogously, if $p = 2$, $q = \infty$, from $c_k(I_n\colon l^n_1 \to l^n_2) = d_k(I_n\colon l^n_2 \to l^n_\infty)$, Ref. [24], (11.7.6), we obtain, again by Theorem A,

$$\begin{aligned} L^{(e)}_{r,\infty}(I_n\colon l^n_2 \to l^n_\infty) &\leqslant \rho_4(r)\, L^{(d)}_{r,\infty}(I_n\colon l^n_2 \to l^n_\infty) \\ &\leqslant \rho_4(r)\, L^{(c)}_{r,\infty}(I_n\colon l^n_1 \to l^n_2) \\ &\leqslant \rho_5(r)\, n^{1/r - 1/2} \end{aligned}$$

for $0 < r < 2$ and $n = 1, 2,\ldots$.

Now we treat the case $p = 1$, $q = \infty$. Let $0 < w < 1$; then we can find an r

with $0<r<2$ and $1/w=2/r=1/r+1/r$. Applying the estimates just obtained we get

$$\begin{aligned} L_{w,\infty}^{(e)}(I_n: l_1^n \to l_\infty^n) &\leqslant \rho_6(w)\, L_{r,\infty}^{(e)}(I_n: l_2^n \to l_\infty^n)\, L_{r,\infty}^{(e)}(I_n: l_1^n \to l_2^n) \\ &\leqslant \rho_7(w)\, n^{1/r-1/2} n^{1/r-1/2} \\ &\leqslant \rho_7(w)\, n^{1/w-1} \end{aligned}$$

for $n=1, 2,\ldots.$

In order to check the general case $1 \leqslant p < q \leqslant \infty$ we use interpolation properties of entropy numbers [24, (12.1.11)].

By [24, p. 173], we have

$$e_k(I_n: l_p^n \to l_q^n) \leqslant 4 e_k(I_n: l_1^n \to l_\infty^n)^{1/p-1/q}.$$

Assume $1/s_0 > 1/p - 1/q$ and put $w := s_0(1/p-1/q)$ (<1). Then

$$\begin{aligned} L_{s_0,\infty}^{(e)}(I_n: l_p^n \to l_q^n) &\leqslant 4 L_{w,\infty}^{(e)}(I: l_1^n \to l_\infty^n)^{w/s_0} \\ &\leqslant \rho_8(s_0, p, q)\, n^{(1/w-1)w/s_0} \\ &\leqslant \rho_8(s_0, p, q)\, n^{1/s_0+1/q-1/p} \end{aligned}$$

for $n=1, 2,\ldots.$

Finally, if $1/s > 1/p - 1/q$ and $0 < t \leqslant \infty$ we choose s_0 such that $1/s > 1/s_0 > 1/p - 1/q$. From Lemma 2 of [4] we obtain the inequality

$$L_{s,t}^{(e)}(I_n: l_p^n \to l_q^n) \leqslant \rho_9(s, s_0, t)\, n^{1/s-1/s_0} L_{s_0,\infty}^{(e)}(I_n: l_p^n \to l_q^n)$$

for $n=1, 2,\ldots,$ and thus by using the preceding estimate we obtain the required assertion

$$L_{s,t}^{(e)}(I_n: l_p^n \to l_q^n) \leqslant \rho(s, t, p, q)\, n^{1/s+1/q-1/p} \qquad \text{for } n=1, 2,\ldots.$$

Remark. Recently Höllig [9] gave a direct proof for

$$e_k(I_n: l_1^n \to l_\infty^n) \leqslant \rho\, \frac{\log^2(n/k+1)}{k}, \qquad 1 \leqslant k \leqslant n.$$

From this estimate we obtain also the satement of Theorem 1 in the case $1 \leqslant p < q \leqslant \infty$.

3. Diagonal Operators between l_p Spaces

In this section we deal with diagonal operators $D \in \mathscr{L}(l_p, l_q)$ defined by $D(\xi_i) := (\sigma_i \xi_i)$ and generated by a sequence (σ_i). In the following we give necessary and sufficient conditions for D to belong to $\mathscr{L}^{(e)}_{s,t}$.

The results given below extend those of Marcus [16], formulated in the language of ε-entropy, and Oloff [21] to the nontrivial and interesting case $1 \leqslant p < q \leqslant \infty$. It turns out that the characterization of diagonal operators by their entropy numbers follows from the behaviour of the entropy quasinorms $L^{(e)}_s(I_n: l^n_p \to l^n_q)$ in Section 2.

We start with a necessary condition.

Proposition 1. *If* $1 \leqslant p,\ q \leqslant \infty,\ 1/r > \max(1/q - 1/p; 0),\ 0 < t \leqslant \infty$, *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$D \in \mathscr{L}^{(e)}_{s,t}(l_p, l_q) \qquad \textit{implies} \qquad (\sigma_i) \in l_{r,t}.$$

Proof. Without loss of generality we may suppose that $|\sigma_1| \geqslant |\sigma_2| \geqslant \cdots \geqslant 0$. If $D \in \mathscr{L}^{(e)}_{s,t}(l_p, l_q)$ and $\sigma_k = 0$ for $k \geqslant m$ then the statement is trivial. So we assume that $|\sigma_1| \geqslant |\sigma_2| \geqslant \cdots > 0$. Define operators $J_n \in \mathscr{L}(l^n_p, l_p)$ and $Q_n \in \mathscr{L}(l_p, l^n_p)$ by

$$J_n(\xi_1, ..., \xi_n) := (\xi_1, ..., \xi_n, 0, 0, ...)$$

and

$$Q_n(\xi_1, ..., \xi_n, \xi_{n+1}, ...) := (\xi_1, ..., \xi_n).$$

Obviously, $\|J_n\| = \|Q_n\| = 1$ and $D_n = Q_n D J_n$ is invertible. Clearly, the identity operator $I_n: l^n_\infty \to l^n_1$ can be factorized as follows:

$$I_n: l^n_\infty \xrightarrow{I_n} l^n_p \xrightarrow{D_n^{-1}} l^n_p \xrightarrow{J_n} l_p \xrightarrow{D} l_q$$
$$\xrightarrow{Q_n} l^n_q \xrightarrow{I_n} l^n_1.$$

From [24], (12, 2.1), we have $e_n(I_n: l^n_\infty \to l^n_1) \geqslant (1/6)\, n$ and thus the multiplicativity of the entropy numbers yields

$$\begin{aligned}
(1/6)\, n &\leqslant e_n(I_n: l^n_\infty \to l^n_1) \\
&\leqslant e_n(I_n Q_n D J_n: l^n_p \to l^n_1)\, \|D_n^{-1} I_n: l^n_\infty \to l^n_p\| \\
&\leqslant \|I_n: l^n_q \to l^n_1\|\, \|Q_n\|\, e_n(D)\, \|J_n\|\, \|I_n: l^n_\infty \to l^n_p\|\, \|D_n^{-1}\| \\
&\leqslant n^{1-1/q} e_n(D: l_p \to l_q)\, n^{1/p}\, |\sigma_n|^{-1}.
\end{aligned}$$

Therefore

$$(1/6)\,|\sigma_n|\, n^{1/p} \leqslant e_n(D\colon l_p \to l_q) \qquad \text{for } n = 1, 2,....$$

Hence

$$\sum_{n=1}^{\infty} n^{t/r-1}\,|\sigma_n|^t \leqslant 6^t \sum_{1}^{\infty} n^{t/r-1} n^{t(1/p-1/q)} e_n^t(D)$$
$$\leqslant 6^t \sum_{1}^{\infty} n^{t/s-1} e_n^t(D) < \infty.$$

This implies $(\sigma_n) \in l_{r,t}$.

Now we prove the sufficiency of the above condition.

PROPOSITION 2. *If* $1 \leqslant p,\ q \leqslant \infty,\ 1/r > \max(1/q - 1/p; 0),\ 0 < t \leqslant \infty,$ *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$(\sigma_i) \in l_{r,t} \qquad \textit{implies} \qquad D \in \mathscr{L}_{s,t}^{(e)}(l_p, l_q).$$

Proof. First we consider the special case where $(\sigma_i) \in l_{r,\infty}$. We may again assume that $|\sigma_1| \geqslant |\sigma_2| \geqslant \cdots \geqslant 0$. Define canonical operators $J^{(k)} \in \mathscr{L}(l_p^{2^k}, l_p)$ and $Q^{(k)} \in \mathscr{L}(l_p, l_p^{2^k})$ by

$$J^{(k)}(\xi_1,..., \xi_{2^k}) := (0; 0, 0;...; \xi_1,..., \xi_{2^k}; 0, 0,...)$$

and

$$Q^{(k)}(\xi_1,..., \xi_n,...) := (\xi_{2^k},..., \xi_{2^{k+1}-1}) \qquad \text{for } k \geqslant 0.$$

Clearly, $\|J^{(k)}\| = \|Q^{(k)}\| = 1$.

In the sequel the identity operator of the 2^k-dimensional vector space is denoted by $I^{(k)}$. Furthermore we define $D^{(k)} \in \mathscr{L}(l_p^{2^k}, l_q^{2^k})$ for $k \geqslant 0$ by

$$D^{(k)}(\xi_1,..., \xi_{2^k}) := (\sigma_{2^k}\xi_1,..., \sigma_{2^{k+1}-1}\xi_{2^k}).$$

Obviously, $D = \sum_{k=0}^{\infty} J^{(k)} D^{(k)} Q^{(k)}$, where $Q^{(k)} \in \mathscr{L}(l_p, l_p^{2^k})$, $D^{(k)} \in \mathscr{L}(l_p^{2^k}, l_q^{2^k})$ and $J^{(k)} \in \mathscr{L}(l_q^{2^k}, l_q)$.

Now, choose s such that $1/s > 1/r + 1/p - 1/q$. Then by Theorem 1 of Section 2 we have

$$L_s^{(e)}(D^{(k)}\colon l_p^{2^k} \to l_q^{2^k}) \leqslant L_s^{(e)}(I^{(k)}\colon l_p^{2^k} \to l_q^{2^k})\, \|D^{(k)}\colon l_p^{2^k} \to l_p^{2^k}\|$$
$$\leqslant \rho_0(s, p, q)\, 2^{k(1/s+1/q-1/p)}\, |\sigma_{2^k}|.$$

Since $L_s^{(e)}$ is equivalent to an $\alpha := \alpha(s)$ ideal norm [24], (6.2.5), it follows, because of $1/s - 1/r + 1/q - 1/p > 0$, that

$$\begin{aligned}
L_s^{(e)}\left(\sum_0^{m-1} J^{(k)}D^{(k)}Q^{(k)}\right) &\leqslant \rho_1(s)\left(\sum_0^{m-1} L_s^{(e)}(J^{(k)}D^{(k)}Q^{(k)})^\alpha\right)^{1/\alpha}\\
&\leqslant \rho_1(s)\left(\sum_0^{m-1} \|J^{(k)}\|^\alpha\, L_s^{(e)}(D_k)^\alpha\, \|Q^{(k)}\|^\alpha\right)^{1/\alpha}\\
&\leqslant \rho_2(s,p,q)\left(\sum_0^{m-1} 2^{k\alpha(1/s+1/q-1/p)}\,|\sigma_{2^k}|^\alpha\right)^{1/\alpha}\\
&\leqslant \rho_3(s,r,p,q)\left(\sum_0^{m-1} 2^{k\alpha(1/s-1/r+1/q-1/p)}\right)^{1/\alpha}\\
&\leqslant \rho_4(s,r,p,q)\,2^{m(1/s-1/r+1/q-1/p)}.
\end{aligned}$$

This implies

$$2^{(m-1)1/s} e_{2m-1}\left(\sum_0^{m-1} J^{(k)}D^{(k)}Q^{(k)}\right) \leqslant \rho_4(s,r,p,q)\,2^{m(1/s-1/r+1/q-1/p)}$$

and, consequently,

$$e_{2m-1}\left(\sum_0^{m-1} J^{(k)}D^{(k)}Q^{(k)}\right) \leqslant \rho_5(r,p,q)\,2^{m(-1/r+1/q-1/p)}.$$

In order to estimate the remainder $\sum_m^\infty J^{(k)}D^{(k)}Q^{(k)}$ we choose s such that $1/r + 1/p - 1/q > 1/s > \max(1/p - 1/q; 0)$. This is possible since $1/r > \max(1/q - 1/p; 0)$. Because of $1/s - 1/r + 1/q - 1/p < 0$ we get

$$\begin{aligned}
&L_s^{(e)}\left(\sum_m^\infty J^{(k)}D^{(k)}Q^{(k)}\right)\\
&\leqslant \rho_6(s)\left(\sum_m^\infty 2^{k\alpha(1/s-1/r+1/q-1/p)}\right)^{1/\alpha}\\
&\leqslant \rho_6(s)\,2^{m(1/s-1/r+1/q-1/p)}\left(\sum_1^\infty 2^{k\alpha(1/s-1/r+1/q-1/p)}\right)^{1/\alpha}\\
&\leqslant \rho_7(s,r,p,q)\,2^{m(1/s-1/r+1/q-1/p)}
\end{aligned}$$

by estimates similar to those used before.

Thus

$$e_{2m-1}\left(\sum_m^\infty J^{(k)}D^{(k)}Q^{(k)}\right) \leqslant \rho_{8(r,p,q)}\,2^{m(-1/r+1/q-1/p)}.$$

The additivity of the entropy numbers yields

$$\begin{aligned} e_{2^m}(D) &\leqslant e_{2m-1}(D) \\ &\leqslant e_{2m-1}\left(\sum_{0}^{m-1} J^{(k)}D^{(k)}Q^{(k)}\right) + e_{2m-1}\left(\sum_{m}^{\infty} J^{(k)}D^{(k)}Q^{(k)}\right) \\ &\leqslant \rho_9(r,p,q)\, 2^{m(-1/r+1/q-1/p)}. \end{aligned}$$

If n is a natural number we take m such that $2^m \leqslant n < 2^{m+1}$. By the monotonicity of the entropy numbers and the preceding estimate we obtain

$$\begin{aligned} e_n(D) \leqslant e_{2^m}(D) &\leqslant \rho_9(r,p,q)\, 2^{m(-1/r+1/q-1/p)} \\ &\leqslant \rho_{10}(r,p,q)\, 2^{(m+1)(-1/r+1/q-1/p)} \\ &\leqslant \rho_{10}(r,p,q)\, n^{-1/r+1/q-1/p}. \end{aligned}$$

Hence

$$D \in \mathscr{L}^{(e)}_{s,\infty}(l_p, l_q) \qquad \text{for } 1/s = 1/r + 1/p - 1/q.$$

Finally, the remaining part can be checked by real interpolation: If $0 < s_0 < s_1 < \infty$, $0 < \Theta < 1$, and $1/s = (1-\Theta)/s_0 + \Theta/s_1$ then

$$(\mathscr{L}^{(e)}_{s_0,t_0}(E,F), \mathscr{L}^{(e)}_{s_1,t_1}(E,F))_{\Theta,t} \subseteq \mathscr{L}^{(e)}_{s,t}(E,F) \qquad \text{for} \quad 0 < t_0, t_1, t \leqslant \infty.$$

The proof of this fact can be carried out in the same way as the proof of Theorem 14 in [25]. Furthermore we use the following classical interpolation result (cf. [1, 27]):

$$(l_{r_0,t_0}, l_{r_1,t_1})_{\Theta,t} = l_{r,t} \qquad \text{for } 0 < r_0 < r_1 < \infty, \quad 0 < \Theta < 1,$$
$$\frac{1}{r} = \frac{1-\Theta}{r_0} + \frac{\Theta}{r_1}, \qquad 0 < t_0, t_1, t \leqslant \infty.$$

Now, given an exponent r with $1/r > \max(1/q - 1/p; 0)$ we can find r_0, r_1 and Θ such that $1/r_0 > 1/r > 1/r_1 > \max(1/q - 1/p; 0)$ and $1/r = (1-\Theta)/r_0 + \Theta/r_1$. As already shown for the operator σ transforming every sequence (σ_k) into the diagonal operator D we have

$$\sigma\colon l_{r_i,\infty} \to \mathscr{L}^{(e)}_{s_i,\infty}(l_p, l_q),$$

where $1/s_i = 1/r_i + 1/p - 1/q$, $i = 0, 1$. Applying the preceding interpolation formulas we obtain

$$\sigma\colon l_{r,t} \to \mathscr{L}^{(e)}_{s,t}(l_p, l_q)$$

with $1/s = (1-\Theta)/s_0 + \Theta/s_1 = 1/r + 1/p - 1/q$. This completes the proof.

Summarizing Propositions 1 and 2 we can establish

THEOREM 2. *Let* $1 \leqslant p, q \leqslant \infty$, $1/r > \max(1/q - 1/p; 0)$, $0 < t \leqslant \infty$, *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$D \in \mathscr{L}_{s,t}^{(e)}(l_p, l_q) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t}.$$

Remarks. As an immediate consequence of Theorem 2 we get the classical result of Mitjagin [17] originally formulated in the language of ε-entropy, which says that the components $\mathscr{L}_{s,t}^{(e)}$ on the Hilbert space coincide with the well-known Schatten classes $\mathscr{S}_{s,t}$; cf. [29].

The results of Marcus [16] and Oloff [21], mentioned above, are contained in the statement of Theorem 2 in the case $\infty \geqslant p \geqslant q \geqslant 1$. The statement of Theorem 2 in the case $1 \leqslant p \leqslant q \leqslant \infty$ possesses interesting applications to eigenvalue problems of operators acting on Banach spaces and also applications in the theory of probability; cf. [8, 16]. In [5], Theorem 2 was also used to characterize embedding maps between Besov spaces by entropy numbers. In this paper the famous results of Birman and Solomjak [2] and Triebel [28] are obtained and extended to the unknown cases in [2, 28].

4. DIAGONAL OPERATORS BETWEEN LORENTZ SEQUENCE SPACES

Under special assumptions the Lorentz sequence spaces can be normed: If $1 < p < \infty$ and $1 \leqslant u \leqslant \infty$ then

$$\|x\|_{p,u} := \left(\sum_1^\infty \left[n^{1/p - 1/u - 1} \sum_{k=1}^n s_k(x)\right]^u\right)^{1/u} \qquad \text{for } 1 \leqslant u < \infty$$

$$:= \sup_{1 \leqslant n < \infty} \left[n^{1/p - 1} \sum_{k=1}^n s_k(x)\right] \qquad \text{for } u = \infty$$

defines an equivalent norm on $l_{p,u}$.

In the sequel we extend the statements of the preceding section to diagonal operators acting from $l_{p,u}$ into $l_{q,v}$, $1 < p, q < \infty$, $1 \leqslant u, v \leqslant \infty$. It turns out that the characterization of diagonal operators belonging to the entropy ideals $\mathscr{L}_{s,t}^{(e)}(l_{p,u}, l_{q,v})$ does not depend on the indices u and v of the underlying Lorentz sequence spaces.

We again start with the necessary condition.

PROPOSITION 3. *Let* $1 < p, \ q < \infty, \ 1 \leqslant u, \ v \leqslant \infty, \ 1/r > \max(1/q - 1/p; 0)$, $0 < t \leqslant \infty$, *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$D \in \mathscr{L}_{s,t}^{(e)}(l_{p,u}, l_{q,v}) \qquad \textit{implies} \qquad (\sigma_i) \in l_{r,t}.$$

The proof of this statement is analogous to that of Proposition 1. Now we turn to the sufficiency of the condition.

PROPOSITION 4. *Let* $1 < p,\ q < \infty,\ 1 \leqslant u,\ v \leqslant \infty,\ 1/r > \max(1/q - 1/p; 0)$, $0 < t \leqslant \infty$, *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$(\sigma_i) \in l_{r,t} \qquad \textit{implies} \qquad D \in \mathscr{L}_{s,t}^{(e)}(l_{p,u}, l_{q,v}).$$

Proof. First we show that $(\sigma_i) \in l_{r,1}$ implies $D \in \mathscr{L}_{s,\infty}^{(e)}(l_{p,\infty}, l_{q,1})$ for $1/r > \max(1/q - 1/p; 0)$ and $1/s = 1/r + 1/p - 1/q$. Since $1/r > \max(1/q - 1/p; 0)$ we can find p_0, q_0 such that $p > p_0$, $q_0 > q$, and $1/r > \max(1/q_0 - 1/p_0; 0)$. By a result of Oloff [21], the operator $D \in \mathscr{L}(l_{p,\infty}, l_{q,1})$ can be factorized with diagonal operators D_0, D_1 and $\tilde{D}$ as

$$\begin{tikzcd} l_{p,\infty} \arrow[r, "D"] \arrow[d, "D_0"'] & l_{q,1} \\ l_{p_0} \arrow[r, "\tilde{D}"] & l_{q_0} \arrow[u, "D_1"'] \end{tikzcd},$$

where the generating sequences of these diagonal operators satisfy the conditions

$$(\sigma_i^{(0)}) \in l_{r_0,p_0}, \qquad \frac{1}{r_0} = \frac{1}{p_0} - \frac{1}{p},$$

$$(\sigma_i^{(1)}) \in l_{r_1,q_0'}, \qquad \frac{1}{r_1} = \frac{1}{q} - \frac{1}{q_0}, \qquad \frac{1}{q_0'} = 1 - \frac{1}{q_0},$$

and

$$(\tilde{\sigma}_i) \in l_{\tilde{r},\tilde{t}}, \qquad \frac{1}{\tilde{r}} = \frac{1}{r} - \frac{1}{r_0} - \frac{1}{r_1}, \qquad \frac{1}{\tilde{t}} = \frac{1}{q_0} - \frac{1}{p_0}.$$

Applying Theorem 2 of the preceding section to the operator $\tilde{D}$ we obtain

$$\tilde{D} \in \mathscr{L}_{s,t}^{(e)}(l_{p_0}, l_{q_0}) \subseteq \mathscr{L}_{s,\infty}^{(e)}(l_{p_0}, l_{q_0})$$

for

$$\frac{1}{s} = \frac{1}{\tilde{r}} + \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{r} - \frac{1}{r_0} - \frac{1}{r_1} + \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

Thus the ideal property of the classes $\mathscr{L}^{(e)}_{s,\infty}$ yields

$$D \in \mathscr{L}^{(e)}_{s,\infty}(l_{p,\infty}, l_{q,1}) \qquad \text{for } \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

Now the preceding result can be improved by real interpolation: Given an exponent r with $1/r > \max(1/q - 1/p; 0)$ we can find r_0, r_1, and Θ such that $1/r_0 > 1/r > 1/r_1 > \max(1/q - 1/p; 0)$ and $1/r = (1-\Theta)/r_0 + \Theta/r_1$. As already shown for the operator σ transforming every sequence (σ_k) into the diagonal operator D we have

$$\sigma: l_{r_i,1} \to \mathscr{L}^{(e)}_{s_i,\infty}(l_{p,\infty}, l_{q,1}),$$

where $1/s_i = 1/r_i + 1/p - 1/q$, $i = 0, 1$. Applying the interpolation formulas given in the proof of Proposition 2 we get

$$\sigma: l_{r,t} \to \mathscr{L}^{(e)}_{s,t}(l_{p,\infty}, l_{q,1})$$

with $1/s = (1-\Theta)/s_0 + \Theta/s_1 = 1/r + 1/p - 1/q$.

The general case, where $(\sigma_i) \in l_{r,t}$, $D \in \mathscr{L}(l_{p,u}, l_{q,v})$, can be checked from the result just obtained using the ideal property of the classes $\mathscr{L}^{(e)}_{s,t}$ and the factorization diagram

$$\begin{array}{ccc} l_{p,u} & \xrightarrow{D} & l_{q,v} \\ {\scriptstyle I}\downarrow & & \uparrow{\scriptstyle I} \\ l_{p,\infty} & \xrightarrow{D} & l_{q,1} \end{array},$$

where I is the identity operator.

Summarizing Propositions 3 and 4 we can establish

THEOREM 3. *Let* $1 < p, q < \infty$, $1 \leqslant u, v \leqslant \infty$, $1/r > \max(1/q - 1/p; 0)$, $0 < t \leqslant \infty$, *and* $1/s = 1/r + 1/p - 1/q$. *Then*

$$D \in \mathscr{L}^{(e)}_{s,t}(l_{p,u}, l_{q,v}) \qquad \textit{if and only if} \qquad (\sigma_i) \in l_{r,t}.$$

5. Eigenvalue Distributions of Factorable Operators

An operator $S \in \mathscr{L}(E, F)$ is called an $(r, t/p, q)$-factorable operator, $0 < r < \infty$, $0 < t \leqslant \infty$, $1 < p, q < \infty$, iff there exists a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{S} & F \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle Y} \\ l_{p,\infty} & \xrightarrow{D} & l_{q,1} \end{array}$$

such that $D \in \mathscr{L}(l_{p,\infty}, l_{q,1})$ is a diagonal operator of the form $D(\xi_i) = (\sigma_i \xi_i)$ with $(\sigma_i) \in l_{r,t}$, $1/r > \max(1/q - 1/p; 0)$, $A \in \mathscr{L}(E, l_{p,\infty})$ and $Y \in \mathscr{L}(l_{q,1}, F)$. The class of these operators is denoted by $\mathscr{F}_{(r,t/p,q)}$.

In this section we apply the preceding results to eigenvalue problems of factorable operators acting on arbitrary (complex) Banach spaces. For $1 < p \leqslant q < \infty$ we show that the sequence of eigenvalues $(\lambda_n(S))$ of operators $S \in \mathscr{F}_{(r,t/p,q)}(E, E)$ belongs to certain Lorentz sequence spaces $l_{s,t}$. In this connection the eigenvalues $\lambda_n(S)$ of S are ordered in nonincreasing absolute values and counted according to their algebraic multiplicities. The results given below complement those obtained in [3, 10, 12] for nuclear operators.

For our purpose we need a further result of [4] (cf. also [7]).

THEOREM C. *Let $S \in \mathscr{L}(E, E)$ be a compact operator on a (complex) Banach space. Then*

$$|\lambda_n(S)| \leqslant \sqrt{2}\, e_{n+1}(S)$$

for $n = 1, 2, \ldots$.

Now we can state

THEOREM 4. *Let $1 < p \leqslant q < \infty$, $0 < r < \infty$, $0 < t \leqslant \infty$, and $1/s = 1/r + 1/p - 1/q$. Then $(\lambda_n(S)) \in l_{s,t}$ for any $S \in \mathscr{F}_{(r,t,p,q)}(E, E)$ and any E.*

Proof. Assume $S \in \mathscr{F}_{(r,t/p,q)}(E, E)$. Then by the definition of these operators we have the factorization diagram

$$\begin{tikzcd} E \arrow[r, "S"] \arrow[d, "A"'] & E \\ l_{p,\infty} \arrow[r, "D"] & l_{q,1} \arrow[u, "Y"'] \end{tikzcd}.$$

By Theorem 3, $D \in \mathscr{L}^{(e)}_{s,t}(l_{p,\infty}, l_{q,1})$ for $1/s = 1/r + 1/p - 1/q$, and thus, $S \in \mathscr{L}^{(e)}_{s,t}(E, E)$. Now, Theorem C yields the desired assertion

$$(\lambda_n(S)) \in l_{s,t} \qquad \text{for } \frac{1}{s} = \frac{1}{r} + \frac{1}{p} - \frac{1}{q}.$$

The conclusion in Theorem 4 is optimal in the sense of

PROPOSITION 5. *Let $1 < p \leqslant q < \infty$, $0 < r < \infty$, $0 < t < \infty$, and $1/s = 1/r + 1/p - 1/q$. Then for any $t_0 < t$ there exists a diagonal operator $D \in \mathscr{F}_{(r,t/p,q)}(l_{p,\infty}, l_{p,\infty})$ such that $(\lambda_n(D)) \notin l_{s,t_0}$.*

Proof. Given a diagonal operator $D\colon l_{p,\infty} \to l_{p,\infty}$ generated by a sequence $(\sigma_k) \in l_{s,t} \backslash l_{s,t_0}$ with $1/s = 1/r + 1/p - 1/q$ and $t_0 < t$. Obviously, $(\lambda_n(D)) \notin l_{s,t_0}$. However, D belongs to $\mathscr{F}_{(r,t/p,q)}(l_{p,\infty}, l_{p,\infty})$ because of

$$\begin{tikzcd} l_{p,\infty} \arrow[rr, "D"] \arrow[dr, "D_1"'] & & l_{p,\infty} \arrow[dl, "D_2"] \\ & l_{q,1} & \end{tikzcd}$$

where D_1 and D_2 are diagonal operators generated by sequences $(\sigma_i^{(1)}) \in l_{r,t}$ and $(\sigma_i^{(2)}) \in l_{v,\infty}$, for $1/v = 1/p - 1/q$. Such a decomposition of D exists according to a result of Oloff [21].

Remark. For a detailed study of eigenvalues of factorable operators in the case $1 < q \leqslant p < \infty$ one has to use the concept of Weyl numbers recently developed by Pietsch [25].

References

1. J. Bergh and J. Löfström, "Interpolation Spaces," Springer-Verlag, Berlin/Heidelberg/NewYork, 1976.
2. M. S. Birman and M. Z. Solomjak, Approximation of functions of the W_p-classes by piecewise-polynomial functions, *Mat. Sb.* **73** (1967), 331–355. [Russian]
3. B. Carl, Distribution of the eigenvalues of nuclear operators in Banach spaces, *Vestnik Moskov Univ. Ser. Mat. Meh.* **31** (1976), 3–11. [Russian]
4. B. Carl, Entropy numbers, s-numbers, and eigenvalue problems, *J. Funct. Anal.* **41** (1981), 290–306
5. B. Carl, Entropy numbers of embedding maps between Besov spaces with an application to eigenvalue problems, *Proc. Royal Soc. Edinburgh Sect. A* **90** (1981).
6. B. Carl and A. Pietsch, Entropy numbers of operators in Banach spaces, *in* "Lecture Notes in Mathematics No. 609," pp. 21–33, Springer-Verlag, Berlin/Heidelberg/New York, 1977.
7. B. Carl and H. Triebel, Inequalities between eigenvalues, entropy numbers and related quantities of compact operators in Banach spaces, *Math. Ann.* **251** (1980), 129–133.
8. R. M. Dudley, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes, *J. Funct. Anal.* **1** (1967), 290–330.
9. K. Höllig, Diameters of classes of smooth functions, preprint, Bonn, 1979.
10. W. B. Johnson, H. König, B. Maurey, and J. R. Retherford, Eigenvalues of p-summing and l_p-type operators in Banach spaces, *J. Funct. Anal.* **32** (1979), 353–380.
11. B. S. Kashin, Diameters of some finite dimensional sets and of some classes of smooth functions, *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (1977), 334–351. [Russian]
12. H. König, J. R. Retherford, and N. Tomczak-Jaegermann, On the eigenvalues of $(p, 2)$-summing operators and constants associated to normed spaces, *J. Funct. Anal.*
13. A. N. Kolmogorov and V. M. Tichomivov, ε-entropy and ε-capacity of sets in function spaces, *Usp. Mat. Nauk.* **14**, No. 2 (1959), 3–86. [Russian]
14. G. G. Lorentz, Metric entropy and approximation, *Bull. Amer. Math. Soc.* **72** (1966), 903–937.

15. G. G. LORENTZ, "Approximation of Functions," New York, 1966.
16. M. B. MARCUS, The ε-entropy of some compact subsets of l^p, *J. Approx. Theory* **10** (1974), 304–312.
17. B. S. MITJAGIN, Approximative dimension and bases in nuclear spaces, *Usp. Mat. Nauk* **16**, No. 4 (1961), 63–132. [Russian]
18. B. S. MITJAGIN, Random matrices and subspaces, *in* Collection of papers on "Geometry of Linear Spaces and Operator Theory," pp. 175–202, Jaroslavl, 1977. [Russian]
19. B. S. MITJAGIN AND A. PEŁCZYŃSKI, Nuclear operators and approximative dimension, *in* "Proceeedings, ICM, Moscow, 1966," pp. 366–372.
20. B. S. MITJAGIN AND V. M. TICHOMIVOV, Asymptotical characteristics of compact sets in linear spaces, *Trudy IV. Vsesojus. Mat. Sezda*, Leningrad, 1961, **2** (1964), 299–308. [Russian]
21. R. OLOFF, Entropieeigenschaften von Diagonaloperatoren, *Math. Nachr.* **86** (1978), 157–165.
22. R. OLOFF, Remarks on Lorentz sequence spaces, preprint, Jena, 1979.
23. J. PEETRE, ε-entropie, ε-capacité et espaces d'interpolation, *Ricerche Math.* **17** (1968), 216–220.
24. A. PIETSCH, "Operator Ideals," Berlin, 1978.
25. A. PIETSCH, Weyl numbers and eigenvalues of operators in Banach spaces, *Math. Ann.* **247** (1980), 149–168.
26. L. S. PONTRIJAGIN AND L. G. SCHNIRELMAN, Sur une propriété métrique de la dimension, *Ann. of Math.* **33** (1932), 156–162.
27. H. TRIEBEL, "Interpolation Theory, Function Spaces, Differential Operatotrs," Berlin, 1978.
28. H. TRIEBEL, Interpolationseigenschaften von Entropie- und Durchmesseridealen kompakter Operatoren, *Studia Math.* **34** (1970), 89–107.
29. H. TRIEBEL, Über die Verteilung der Approximationszahlen kompakter Operatoren in Sobolev–Besov–Räumen, *Invent. Math.* **4** (1967), 275–293.
30. H. TRIEBEL, Interpolation properties of ε-entropy and diameters. Geometrical characteristics of the embedding of function spaces of Sobolev–Besov type, *Mat. Sb.* **98** (1975), 27–41. [Russian]